On the generation of sound by supersonic turbulent shear layers

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A theory is proposed to describe the generation of sound by turbulence at high Mach numbers. The problem is formulated most conveniently in terms of the fluctuating pressure, and a convected wave equation (2.8) is derived to describe the generation and propagation of the pressure fluctuations.

The supersonic turbulent shear zone is examined in detail. It is found that, at supersonic speeds, sound is radiated as eddy Mach waves, and as the Mach number increases, this mechanism of generation becomes increasingly dominant. Attention is concentrated on the properties of the pressure fluctuations just outside the shear zone where the interactions among the weak shock waves have had little effect. An asymptotic solution for large M is derived by a Green's function technique, and it is found that radiation with given frequency n and wave-number $\mathbf{k}$ can be associated with a corresponding critical layer within the shear zone.

It is found that $\overline{(p-p_0)^2}$ increases approximately as $M^{\frac{3}{2}}$ for $M \gg 1$ contrasting with the M^8 variation found by Lighthill for $M \ll 1$. The acoustic efficiency thus varies as $M^{-\frac{3}{2}}$ for $M \gg 1$, and as M^5 for $M \ll 1$, indicating a maximum acoustic efficiency for Mach numbers near one. The directional distribution of the radiation is discussed and the direction of maximum intensity is shown to move towards the perpendicular to the shear zone as M increases. The predictions of the theory are supported qualitatively by the few available experimental observations.

1. Introduction

The problem of acoustic radiation from turbulence has excited interest in recent years because of its relevance to questions of structural fatigue and human discomfort in high-speed aerodynamics. The problem was formulated by Lighthill (1952, 1954), and most subsequent work has been concerned with the application and extension of his technique to various flows at Mach numbers much less than one. A central point in Lighthill's formulation is the emergence of an exact analogy with classical acoustics. From the momentum and continuity equations of the fluid, he showed that the fluctuations in density ρ are described by the equation

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{1.1}$$

where a_0 is the speed of sound in the undisturbed medium and

$$T_{ij} = \rho u_i u_j + p_{ij} - a_0^2 \rho \delta_{ij}. \tag{1·2}$$

In this expression u_i represents the Cartesian velocity components, p_{ij} the stress tensor and δ_{ij} the Kronecker delta tensor.

If the right-hand side of (1.1) were zero, it would be the simple wave equation for the propagation of density fluctuations in a uniform acoustic medium. The full equation is identical with the equation governing sound propagation in an acoustic medium containing a distribution of quadrupole sources whose strength per unit volume is T_{ij}. It follows that there is an exact analogy between the density fluctuations in a real fluid in arbitrary motion and those in a uniform acoustic medium at rest; all the real fluid effects such as sound generation, convection, attenuation and the like are represented by a suitably chosen distribution of acoustic quadrupoles. This acoustical analogy has been powerful in the development of the subject, since it implies that we can use the terminology and concepts of classical acoustics, and the problem really reduces to finding from considerations of the flow dynamics the appropriate distribution $T_{ij}(\mathbf{x}, t)$.

If the Mach number of the flow is small, and this is the case with which most contributions have been concerned, T_{ij} can be approximated by $\rho_0 u_i u_j$; and a retarded potential solution of (1.1) can be written down. This ultimately leads to Lighthill's U^8 law for the intensity of the radiated sound when the Mach number $M \ll 1$. The restriction to low Mach numbers, though not inherent in the original analogy, becomes very important in the course of this analysis. It is involved first in the neglect of $\partial^2(p_{ij} - a_0^2 \rho \delta_{ij})/\partial x_i \partial x_j$ in (1.1) which corresponds physically to the neglect of the sound attenuation and of the variation in the speed of sound at different points in the flow. It is invoked again to replace the density factor in $\rho u_i u_j$ by ρ_0, the density of the undisturbed fluid. This is a good approximation if $(\rho - \rho_0)/\rho_0$ is small, but at high Mach numbers this may not be so. Without this approximation, the retarded potential 'solution' of (1.1) is an integral equation for ρ of a rather intractible kind. This alone would provide a considerable obstacle to extending the Lighthill formulation to high Mach numbers, but it would not be the only one. If $M \gg 1$, the lifetime of an eddy is comparable with, or shorter than, the time taken by a sound pulse to move across the eddy, and the retarded time effect could not at any stage be neglected. Despite this cascade of difficulties, attempts have been made by Ribner (in an unpublished note) and Lilley (1958) to extend the Lighthill approach to mean velocity Mach numbers of about 1·5 or 2, but this seems to be the limit. It has frequently been speculated that, for very large M, the acoustic efficiency might become constant, so that the sound intensity would be proportional to U^3. The author has been unable to find convincing reasons for this belief; it seems that the only reasonable statement of this kind that one can make is that the acoustic efficiency must be bounded above, so that the exponent of U must be less than or equal to 3.

It appears that if we are to be concerned with high Mach numbers, a new approach will be necessary, involving a reformulation of the problem from the beginning. An attempt towards this is made in the present paper. This will involve a reluctant abandonment of the acoustical analogy, and a discussion of the phenomenon as a flow problem at large Mach numbers.

It might be worth while at this point to set forth the detailed aims of this investigation. The first, of course, is to uncover the mechanism whereby sound is generated in a supersonic turbulent flow; it is found to be different from the one that dominates when $M \ll 1$. Secondly, we wish to find how the intensity of the sound field, its directional distribution and the radiated energy flux vary with the appropriate flow parameters, particularly the Mach number M. Associated with this is the question of radiation damping—does the loss of energy from the turbulence by radiation ever become such a large fraction of the energy supply that the turbulence itself tends to be damped out at very large Mach numbers? With these objectives in mind we propose to investigate the sound radiation from a turbulent shear zone, this particular flow being chosen for reasons of relative simplicity and convenience, and in the hope that the results might be experimentally verifiable in, say, the shear zones of a jet.

My first attempt to formulate the problem was in terms of density fluctuations, as in Lighthill's approach. This led to what can best be described as a convected wave equation rather similar to (2.8) below, in which the dependent variable was $\log(\rho/\rho_0)$. This overcame one of the difficulties in the Lighthill formulation in that the density did not occur explicitly on the right-hand side as it does in (1.1). However, the right-hand side consisted of two large terms whose relative importance was difficult to estimate—one involving vorticity fluctuations and one entropy fluctuations. This might indeed have been inferred *a priori* from the work of Chu & Kovasznay (1958) concerning the interactions among the various modes of motion at supersonic speeds. However, if the problem is expressed in terms of pressure fluctuations, this difficulty disappears and the right-hand side of our governing wave equation, (2.8) below, contains only a single dominant term, since pressure-entropy interactions are small.

2. The convected wave equation

In the present discussion, it will be assumed that the effects of gas dissociation and relaxation phenomena can be neglected. These approximations will probably be adequate in aeronautical problems (except in strong shock waves) if the stagnation temperature does not exceed about 2000° K. The equation of state for a perfect gas is

$$p = \mathscr{R}\rho T, \tag{2.1}$$

where $\mathscr{R}$ is the gas constant and p, ρ, T represent the pressure, density and temperature of the gas. In a fixed Cartesian reference frame, the equation of continuity is

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

and since the density can be considered a function of pressure and entropy S, this equation can be expressed alternatively as

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_S\frac{Dp}{Dt}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_p\frac{DS}{Dt}+\frac{\partial u_i}{\partial x_i} = 0. \tag{2.3}$$

From the second law of thermodynamics,

$$dS = c_v\frac{dT}{T}+\left(\frac{\partial p}{\partial T}\right)_\rho d\left(\frac{1}{\rho}\right)$$

(Howarth, 1953, p. 45), where c_v is the specific heat at constant volume. Therefore

$$\begin{aligned}\left(\frac{\partial S}{\partial \rho}\right)_p &= \frac{c_v}{T}\left(\frac{\partial T}{\partial \rho}\right)_p - \frac{1}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho \\ &= -\rho^{-1}(c_v + \mathscr{R}) \\ &= -\frac{c_p}{\rho},\end{aligned} \tag{2.4}$$

from (2.1) and the equation $c_p - c_v = \mathscr{R}$, where c_p is the specific heat at constant pressure. Furthermore,

$$\left(\frac{\partial p}{\partial \rho}\right)_S = a^2 = \frac{\gamma p}{\rho},$$

where a is the local speed of sound and γ the ratio of specific heats, so that (2.3) can be expressed as

$$\frac{\partial u_i}{\partial x_i} = -\frac{1}{\gamma p}\frac{Dp}{Dt} + \frac{1}{c_p}\frac{DS}{Dt}. \tag{2.5}$$

The momentum equation for the fluid is

$$\frac{Du_i}{Dt} = \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j}, \tag{2.6}$$

where p_{ij} represents the stress tensor which, for a Stokesian fluid, is of the form

$$p_{ij} = -p\delta_{ij} + \mu(e_{ij} - \tfrac{2}{3}\theta\delta_{ij}), \tag{2.7}$$

where δ_{ij} is the Kronecker delta, e_{ij} the rate of strain tensor and $\theta = \partial u_i/\partial x_i$ the fluid dilatation.

We now eliminate the linear velocity term between (2.5) and (2.6). Since

$$\frac{\partial}{\partial x_i}\frac{D}{Dt} \equiv \frac{D}{Dt}\frac{\partial}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_j},$$

it follows from (2.6) and (2.7) that

$$\frac{D}{Dt}\frac{\partial u_i}{\partial x_i} = -\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left\{\frac{1}{\rho}\frac{\partial}{\partial x_j}[\mu(e_{ij} - \tfrac{2}{3}\theta\delta_{ij})]\right\}.$$

Operating on equation (2.5) with D/Dt, and subtracting the result from the last expression, we have

$$\begin{aligned}&\frac{D^2}{Dt^2}\log\left(\frac{p}{p_0}\right) - \frac{\partial}{\partial x_i}\left\{a^2\frac{\partial}{\partial x_i}\log\left(\frac{p}{p_0}\right)\right\} \\ &\quad = \gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \gamma\frac{D}{Dt}\left(\frac{1}{c_p}\frac{DS}{Dt}\right) - \gamma\frac{\partial}{\partial x_i}\left\{\frac{1}{\rho}\frac{\partial}{\partial x_j}[\mu(e_{ij} - \tfrac{2}{3}\theta\delta_{ij})]\right\},\end{aligned} \tag{2.8}$$

since

$$\frac{1}{p}\frac{Dp}{Dt} = \frac{D}{Dt}\log\left(\frac{p}{p_0}\right); \quad \frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{a^2}{\gamma}\frac{\partial}{\partial x_i}\log\left(\frac{p}{p_0}\right),$$

where p_0 is a convenient reference pressure.

Equation (2.8) provides the starting point of the present investigation. The terms on the left-hand side are those of a wave equation in a moving medium with a variable speed of sound, the partial time derivatives of the ordinary wave equation being replaced by derivatives following the motion. The first term on the right-hand side represents the generation of pressure fluctuations by the velocity fluctuations in the fluid while the remaining terms describe the effects of entropy fluctuations and fluid viscosity. In contrasting these equations with those of Lighthill, it is evident that the effects of convection and variation in the local speed of sound are included here in the left-hand side of the equations, whereas in Lighthill's acoustic analogy, they were described in terms of a quadrupole distribution in an acoustic medium. In supersonic flow, neither of these effects can reasonably be ignored, whereas if $M \ll 1$, the variation of a^2 is likely to be small, and the convection effects can if necessary be included by a subsequent translation of our frame of reference. In both this and Lighthill's case, the direct effects of fluid viscosity and heat conductivity are likely to be unimportant as far as the sound generation is concerned, and under these circumstances the last two terms of (2.8) can be neglected, since

$$\rho T \frac{DS}{Dt} = \Phi + \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial T}{\partial x_i}\right), \tag{2.9}$$

where Φ is the rate of energy dissipation per unit volume by viscosity and κ the fluid conductivity.

If we were to be concerned with the *structure* of the shock waves developed inside a turbulent fluid at high Mach numbers, then these terms certainly could not be neglected, since the internal structure of the shocks is determined by the balance between these diffusive and the convective effects. For a turbulent shear layer, however, there is the possibility that we might consider separately the processes of generation of the pressure waves and the subsequent development of the shocks. The pressure waves are generated inside the shear zone and they develop into weak or strong shock waves with increasing distance from the turbulent zone. The distance that the wave travels before the shock is fully developed is $\lambda p_0/p'$ (Lighthill 1956, § 5·2), where λ is the wavelength, p_0 the mean pressure and p' the root-mean-square fluctuation in pressure associated with the wave. If this distance is comparable with, or larger than, the shear zone thickness, a separation of the two processes is possible, and the diffusive effects can be neglected as far as the generation is concerned. We would expect that a theory based on this separation would reliably predict the pressure field just outside the shear zone for all wavelengths if p'/p_0 is small and for the larger wavelengths of the sound field if p'/p_0 is moderate.

3. The Fourier transform equation

Consider a turbulent shear zone in which the mean velocity $\overline{u_1}$ is a function of the x_3 position co-ordinate only. Suppose that the characteristic zone thickness is $2L$ and that $\overline{u_1}$ changes from $-U$ to U as x_3 moves from $-\infty$ to ∞. We seek to derive properties of the radiated pressure waves in terms of the velocity fluctuations of the shear zone, our primary concern being with values of U greater than a_0, the speed of sound at infinity. For the sake of analytical convenience we suppose further that all mean-point properties of the motion in the shear zone are independent of x_1, x_2 and time t, being functions of x_3 alone, the position co-ordinate normal to the zone.

Neglecting the diffusive terms in (2.8), we obtain

$$\left\{\frac{D^2}{Dt^2}-\frac{\partial}{\partial x_i}a^2\frac{\partial}{\partial x_i}\right\}\log\left(\frac{p}{p_0}\right)=\gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{3.1}$$

This equation† can be further simplified if we neglect on the left-hand side products of two fluctuating quantities with zero means, on the understanding that these will be small (in mean square) compared with the products of mean and fluctuating quantities. This neglects such physical processes as the convection and scattering of sound by the *turbulence*, and by variation of a^2 about its local mean value. We retain, however, besides the generation term, the processes of convection and refraction of the sound by the mean flow and by variation with x_3 of the local mean speed of sound. The latter may be important if there are significant variations in the mean temperature of the flow. Equation (3.1) then reduces to

$$\left\{\left(\frac{\partial}{\partial t}+\bar{u}_1\frac{\partial}{\partial x_1}\right)^2-\frac{\partial}{\partial x_i}\overline{a^2}\frac{\partial}{\partial x_i}\right\}\log\left(\frac{p}{p_0}\right)=\gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}, \tag{3.2}$$

or

$$\left\{\left(\frac{\partial}{\partial t}+\bar{u}_1\frac{\partial}{\partial x_1}\right)^2-\frac{\partial\overline{a^2}}{\partial x_3}\frac{\partial}{\partial x_3}-\overline{a^2}\frac{\partial^2}{\partial x_i^2}\right\}\log\left(\frac{p}{p_0}\right)=\gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}, \tag{3.3}$$

since $\overline{a^2}=\overline{a^2}(x_3)$. In many instances, the dominant term on the right-hand side is likely to result from the interactions of the large mean velocity gradient with the fluctuating velocity gradients, and the generation term can be approximated by

$$2\gamma\frac{\partial\bar{u}_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}, \tag{3.4}$$

but for the present, the full expression will be retained.

It is of advantage to express (3.3) in dimensionless form. Let the instantaneous and mean velocities be given by

$$v_i=\frac{u_i}{U},\quad V=\bar{v}_1, \tag{3.5a}$$

and also

$$y_i=\frac{x_i}{L},\quad \tau=\frac{tU}{L},$$

$$A^2(y_3)=\frac{\overline{a^2}}{a_0^2},\quad M=\frac{U}{a_0}, \tag{3.5b}$$

where a_0 is a reference sound speed, say at $y_3=+\infty$ in the undisturbed fluid. If the mean stagnation temperature is the same on both sides of the shear zone, then

$$A(y_3)\to 1\quad \text{as}\quad y_3\to\pm\infty; \tag{3.6}$$

the limit as $y_3\to-\infty$ may be different otherwise.

† Since diffusive effects are neglected in this equation, $p\propto\rho^\gamma$ and (3.1) takes the alternative form

$$\left\{\frac{D^2}{Dt^2}-\frac{\partial}{\partial x_i}a^2\frac{\partial}{\partial x_i}\right\}\log\left(\frac{\rho}{\rho_0}\right)=\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i},$$

from which the familiar Lighthill form can be deduced immediately when $M\ll 1$.

The dimensionless form of (3.3) is thus

$$\left\{\left(\frac{\partial}{\partial \tau}+V\frac{\partial}{\partial y_1}\right)^2-\frac{1}{M^2}\frac{dA^2}{dy_3}\frac{\partial}{\partial y_3}-\frac{A^2}{M^2}\frac{\partial^2}{\partial y_i^2}\right\}\log\left(\frac{p}{p_0}\right)=\gamma\frac{\partial v_i}{\partial y_j}\frac{\partial v_j}{\partial y_i}, \tag{3.7}$$

which can be reduced to the normal form for y_3 (in which there is no first differential operator $\partial/\partial y_3$) by taking a new dependent variable

$$\zeta(\mathbf{y},t)=A(y_3)\log\left(\frac{p}{p_0}\right). \tag{3.8}$$

This yields

$$\left\{\left(\frac{\partial}{\partial \tau}+V\frac{\partial}{\partial y_1}\right)^2+\frac{A}{M^2}\frac{d^2A}{dy_3^2}-\frac{A^2}{M^2}\frac{\partial^2}{\partial y_i^2}\right\}\zeta(\mathbf{y},\tau)=\gamma A(y_3)\frac{\partial v_i}{\partial y_j}\frac{\partial v_j}{\partial y_i}$$
$$=G(\mathbf{y},\tau),\quad\text{say.} \tag{3.9}$$

The generation term G is significant only inside the turbulent shear zone, so that

$$G(\mathbf{y},\tau)\to 0\quad\text{as}\quad y_3\to\pm\infty. \tag{3.10}$$

Let us now define the generalized Fourier transforms (Lighthill 1958)

$$\left.\begin{aligned}\zeta(\mathbf{y},\tau)&=\iint\varpi(y_3,\mathbf{k},n)\exp\{i(\mathbf{k}\,.\,\mathbf{y}+n\tau)\}\,d\mathbf{k}\,dn,\\ G(\mathbf{y},\tau)&=\iint\Gamma(y_3,\mathbf{k},n)\exp\{i(\mathbf{k}\,.\,\mathbf{y}+n\tau)\}\,d\mathbf{k}\,dn,\end{aligned}\right\} \tag{3.11}$$

where $\mathbf{k}=(k_1,k_2)$ is a wave-number vector in the plane of the shear layer and the integrations are made over all values of $\mathbf{k}$ and frequency n.

The equation relating the generalized Fourier transforms ϖ and Γ that corresponds to (3.9) is

$$\frac{d^2}{dy^2}\varpi(y,\mathbf{k},n)+\left\{\frac{M^2}{A^2}(n+Vk_1)^2-k^2-\left(\frac{A''}{A}\right)\right\}\varpi(y,\mathbf{k},n)=-\frac{M^2}{A^2}\Gamma(y,\mathbf{k},n), \tag{3.12}$$

where the suffix on y_3 has been dropped, $k^2=k_1^2+k_2^2$, A and V are functions of y only, and $A''=d^2A/dy^2$.

4. Some simple deductions

Before we embark on a detailed analysis of equation (3.12), it might be worth while to discuss briefly the nature of the solutions that we expect in the region $|y|>1$ outside the shear zone. For $y>1$, $G(y)\simeq 0$, $A(y)\simeq 1$, $V(y)\simeq 1$, so that (3.12) becomes

$$\frac{d^2\varpi}{dy^2}+\{M^2(n+k_1)^2-k^2\}\varpi=0. \tag{4.1}$$

Since $\varpi(y)$ must be bounded as $y\to\infty$, the solutions to (4.1) must be either exponentially decreasing or oscillatory according as the coefficient of ϖ is negative or positive. Clearly the oscillatory solution corresponds to a radiated pattern of Mach waves, which for a given Mach number are associated with wave-numbers $\mathbf{k}$ and frequencies n such that

$$M^2(n+k_1)^2-k^2>0. \tag{4.2}$$

Let us consider the contribution to the radiated pressure field from the layer between $y = Y$ and $y = Y + \delta Y$ in the shear zone. The turbulent eddies in this neighbourhood are convected by the mean stream with velocity V_c, say, which is approximately equal to the local mean velocity at Y. If the flow is supersonic, it is sufficient to neglect for the moment the evolution of the eddy pattern as it is carried along, so that the frequency of the components of wave-number $\mathbf{k}$ is just the frequency with which the wave component is swept past the fixed observation point at a rigid convected pattern.† We therefore have

$$n = -k_1 V_c. \tag{4.3}$$

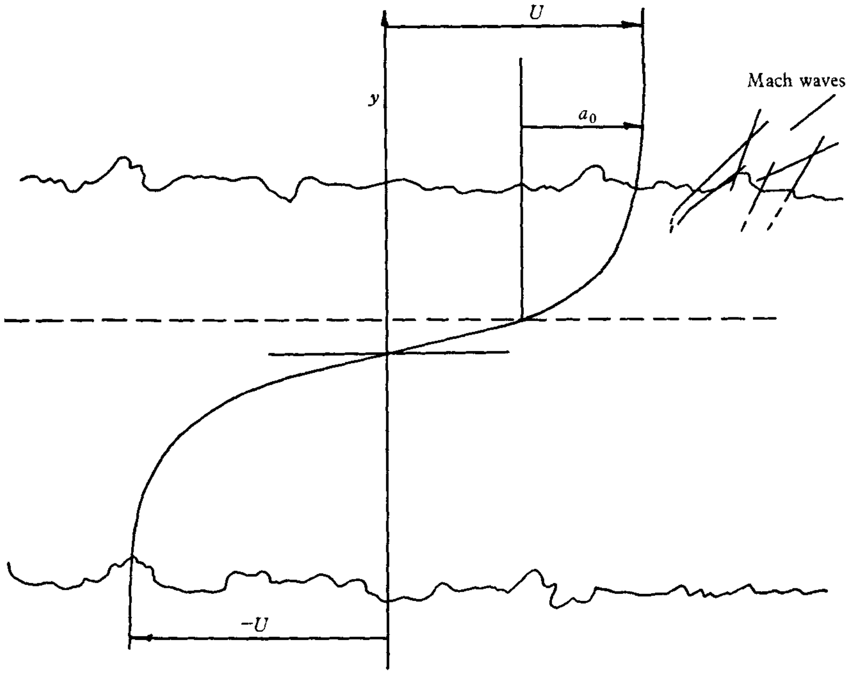

FIGURE 1. The turbulent shear zone.

The layer of the shear zone near Y, where the convection velocity is V_c, will therefore generate Mach waves on the side of the layer $y > 1$ with wave-numbers $\mathbf{k}$ such that

$$M^2 k_1^2 (1 - V_c)^2 > k^2.$$

If θ is the angle between the vector wave-number $\mathbf{k}$ and the direction of the mean velocity, then $\cos\theta = k_1/k$ and this condition becomes

$$\cos^2\theta > [M(1 - V_c)]^{-2}. \tag{4.4}$$

Radiated Mach waves will therefore be generated by some wave-numbers in those layers of the shear zone for which the difference between the mean velocity of the fluid outside the shear zone and the local eddy convection velocity is greater than the speed of sound outside the zone. Generation of Mach waves

† Notice that if the eddy convection velocity is subsonic, this approximation is much too coarse. As Lighthill (1952, 1954) has shown, sound is radiated in subsonic flow *only because* the convected eddy pattern does change as it is carried along.

in this way is somewhat analogous to their formation by thin bodies moving supersonically; they can be described conveniently as 'eddy Mach waves'. This is illustrated in figure 1, where the area below the dashed line, including the whole of the turbulent region in $y < 0$, represents the part of the shear zone which make some contribution to the eddy Mach waves in the fluid above the zone. Clearly, there is a corresponding region towards the upper side of the shear zone that is responsible for Mach wave generation in the fluid below the zone. The possibility that this effect might be important was suggested by Kramer (1955) among others, but hitherto no detailed analysis has been carried out.

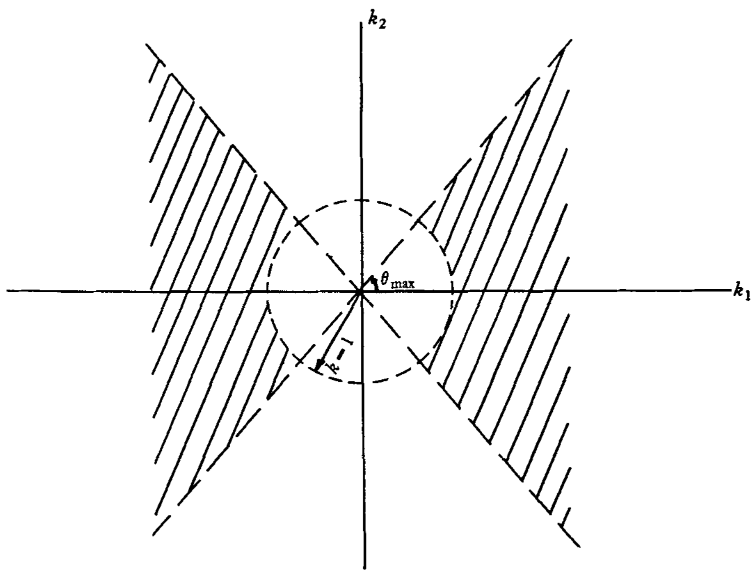

FIGURE 2. Wave-numbers radiating Mach waves from a given critical layer in the shear zone.

For a given layer where the convection velocity is V_c, the allowable wave-numbers for Mach wave generation are given by (4.4) and are indicated in figure 2 by the region inside the lines

$$\theta_{max} = \cos^{-1}\{M(1-V_c)\}^{-1}.$$

However, the maximum eddy size in the shear layer is of order $6L$, or a little more, so that the smallest (dimensionless) wave-numbers present in the turbulence are of order 1 or a little less. We would therefore expect to find the wave-numbers of the radiated pressure field predominantly in the region $\theta < \theta_{max}$, $k > 1$. As the Mach number increases, the critical angle θ_{max} increases also, until at very high Mach numbers, almost all components of the turbulence are capable of radiating Mach waves.

The Fourier components whose wave-numbers lie at angles greater than θ_{max} (whose convection velocities resolved in the direction of $\mathbf{k}$ are subsonic) are still of course capable of radiating some sound since their amplitides and phases change slowly as they are convected. Their contribution could be estimated in a manner analogous to that used by Lighthill (1952). However, if the Mach number of the flow is substantially greater than one, their contribution is probably unimportant, and decreases with increasing M, since for most of the shear layer only a small part of the wave-number plane is excluded by the condition (4.4).

These geometrical considerations do not allow any quantitative predictions concerning the over-all intensity of the radiated field or its distribution among the wave-numbers indicated in figure 2. Such questions can be answered only by a more detailed analysis of (3.12).

5. The radiated pressure field

Equation (3.12) can be expressed as

$$\frac{d^2\varpi(y)}{dy^2}+\left\{M^2q^2(y)-k^2-\left(\frac{A''}{A}\right)\right\}\varpi(y)=-\frac{M^2}{A^2}\Gamma(y), \quad (5.1)$$

where
$$q(y)=\frac{n+k_1V(y)}{A(y)}. \quad (5.2)$$

If $|n|<|k_1|$,† then $q(y)$ passes through zero at some value of y, say Y such that

$$n+k_1V(Y)=0. \quad (5.3)$$

This equation defines the critical layer for a given n and $\mathbf{k}$ more formally than was possible by the intuitive arguments of the previous section, and it will be found that the radiation of frequency n and wave-number $\mathbf{k}$ does indeed originate from the region of the shear zone near Y. When M is large, the coefficient of the term containing $\varpi(y)$ has a small negative region (the subsonic region) near $y=Y$ where $q(Y)=0$. The two zeros are closely spaced, their separation being of order M^{-1}, and the coefficient becomes constant in y as $|y-Y|\to\infty$. This behaviour suggests that we transform (5.1) to the form

$$\frac{d^2\xi}{dz^2}+b^2z^2\xi=f(z), \quad (5.4)$$

where b is a large parameter (of order M) and b^2z^2 has a double zero at $z=0$ corresponding to the closely spaces zeros in $M^2q^2(y)-k^2-(A''/A)$ as $M\to\infty$.

Unless the mean speed of sound changes abruptly within the shear zone (which is unlikely, because of the mixing effect of the turbulence) the term A''/A in this coefficient is small compared with k^2 for the wave-numbers greater than unity with which we are concerned. The principal influence of the variation in the speed of sound in the shear zone has been accounted for by the change of variable (3.8), and even in an extreme case where A changes by a factor of two across the shear zone, the influence of the term A''/A will be felt only by the pressure fluctuations of the largest scale, for which k is of order unity. In general, then, it is sufficient to approximate the coefficient of the second term on the left-hand side of (5.1) by
$$M^2q^2(y)-k^2.$$

† It is clear that there will exist such frequencies in the sound field, since these correspond to the frequencies in the turbulence.

The details of finding the asymptotic solution as $M \to \infty$ to (5.1) are given in the Appendix, but a brief indication of the procedure followed might be in order. Equation (5.1) is transformed so that its left-hand side resembles that of (5.4). One's first inclination would be to seek an asymptotic solution for $-\infty < y < \infty$ in powers of M^{-1}, but it transpires that the first term in this series diverges. If, instead, the domain is divided into the regions $-\infty < y \leqslant Y$ and $Y \leqslant y < \infty$, a convergent asymptotic solution can be found if the large parameter corresponding to b above is taken as

$$\left.\begin{aligned} \lambda &= \left\{M^2 - \frac{k^2}{Q_-^2}\right\}^{\frac{1}{2}} \quad \text{for} \quad -\infty < y \leqslant Y, \\ \mu &= \left\{M^2 - \frac{k^2}{Q_+^2}\right\}^{\frac{1}{2}} \quad \text{for} \quad Y \leqslant y < \infty, \end{aligned}\right\} \tag{5.5}$$

where

$$Q_+ = \lim_{y\to\infty} q(y) = n + k_1,$$

$$Q_- = \lim_{y\to-\infty} q(y) = \frac{n-k_1}{A(-\infty)}.$$

The asymptotic solutions are found in these two domains by a Green's function technique, and the arbitrary constants are determined by applying radiation conditions outside the shear zone and matching conditions at $y = Y$.

In the region outside the shear zone, $y > 1$, the asymptotic expression for the Fourier transform $\varpi(y, \mathbf{k}, n)$ is given from (A. 31), (A. 40), (A. 54) and (A. 57) as

$$\varpi(y, \mathbf{k}, n) \sim \frac{(-\frac{3}{4})!\exp(\frac{3}{8}\pi i)}{4\mu^{\frac{5}{4}}Q_+^{\frac{1}{2}}}\left\{\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} + i\right\} h(0)\exp\left\{-i\mu\int_Y^y q(y)\,dy\right\}, \tag{5.6}$$

for large λ and μ where

$$\left.\begin{aligned} h(0) &= -\frac{M^2\Gamma(Y, \mathbf{k}, n)}{A^2(Y)\,\Omega^{\frac{3}{4}}}, \\ \Omega &= \left(\frac{dV}{dy}\right)_Y, \end{aligned}\right\} \tag{5.7}$$

from (A. 16) and (A. 34). On the other side of the shear zone, $y < -1$, the corresponding expression is

$$\varpi(y, \mathbf{k}, n) \sim \frac{-i(-\frac{3}{4})!\exp(-3\pi i/8)}{4\lambda^{\frac{5}{4}}Q_-^{\frac{1}{2}}}\left\{\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} - i\right\} h(0)\exp\left\{i\lambda\int_Y^y q(y)\,dy\right\}. \tag{5.8}$$

Several properties of these solutions are of interest at this stage. The y-dependence, which appears in the term

$$\exp\left\{-i\mu\int_Y^y q(y)\,dy\right\}$$

in (5.6), and the corresponding term in (5.8) indicates that the solution is of the form of pressure waves, as we would expect, and since $q(y)$ is constant outside the shear layer, the wavelength in the y-direction is $2\pi(\mu Q_+)^{-1}$ for $y > 1$ and $2\pi(\lambda Q_-)^{-1}$ for $y < -1$. Secondly, the radiated pressure fluctuations for a given wave-number and frequency are dependent only on the properties of the turbulence at the critical layer Y appropriate to this $\mathbf{k}$ and n. This provides an analytical confirmation of the intuitive remarks of the previous section. Thirdly, if λ and μ are comparable, that is, if $M^2 \gg k^2/Q_+^2$, k^2/Q_-^2 the ratio of the amplitudes of the pressure waves above and below the shear layer is approximately $(Q_-/Q_+)^{\frac{1}{2}}$. If for a particular pair $(\mathbf{k}, n)$, the critical layer is towards the upper side of the shear zone, then it can be shown simply that $Q_+ < Q_-$, and the amplitude of the pressure wave above the shear zone is greater than the amplitude below for the same $(\mathbf{k}, n)$.

These expressions simplify considerably if $M^2 \gg k^2/Q_+^2$, k^2/Q_-^2, that is if the wave-number of the disturbance lies well within the fan of figure 2. When M is very large, this requirement is satisfied by almost all wave-numbers of the pressure field except those for which k_1 is very small. Then

$$\lambda \simeq \mu \simeq M, \tag{5.9}$$

the errors being of order $\{M \cos^2\theta[1-V(Y)]^2\}^{-1}$, and (5.6) and (5.8) become

$$\left.\begin{aligned} \varpi(y,\mathbf{k},n) &\sim -(-\tfrac{3}{4})!\,(1+i)\exp(3\pi i/8)\,\frac{M^{\frac{3}{4}}\Gamma(Y)}{4Q_+^{\frac{1}{2}}\,A^2(Y)\,\Omega^{\frac{3}{4}}}\exp\left\{-i\mu\int_Y^y q\,dy\right\}, \\ &\text{for } y > 1, \text{ and} \\ \varpi(y,\mathbf{k},n) &\sim (-\tfrac{3}{4})!\,(1+i)\exp(-3\pi i/8)\,\frac{M^{\frac{3}{4}}\Gamma(Y)}{4Q_-^{\frac{1}{2}}\,A^2(Y)\,\Omega^{\frac{3}{4}}}\exp\left\{i\lambda\int_Y^y q\,dy\right\}, \end{aligned}\right\} \tag{5.10}$$

for $y < -1$, where Y is defined by

$$n + k_1 V(Y) = 0.$$

It is convenient now to use the approximation (3.4) for the generation term in the equations and to replace the term G in (3.9) by

$$2\gamma A(y)\frac{\partial V}{\partial y}\frac{\partial v_3}{\partial y_1} = 2\gamma A(y)\,\Omega(y)\frac{\partial v_3}{\partial y_1}.$$

The Fourier transform $\Gamma(y,\mathbf{k},n)$ is thus

$$\Gamma(y,\mathbf{k},n) = -2i\gamma A(y)\,\Omega(y)\,k_1 Z(y,\mathbf{k},n), \tag{5.11}$$

where $Z(y,\mathbf{k},n)$ is the generalized Fourier transform of v_3, the dimensionless velocity fluctuation in the direction normal to the shear zone. From these relations we can synthesize expressions for the spectrum and mean square amplitude of the pressure fluctuation function $\zeta = A(y)\log(p/p_0)$ in terms of the properties of the shear zone. From the theory of generalized functions (Lighthill, 1958) it can be shown that

$$\overline{\varpi(y,\mathbf{k},n)\,\varpi^*(y,\mathbf{k}-\mathbf{k}',n-n')} = \Pi(y,\mathbf{k},n)\,\delta(k_1')\,\delta(k_2')\,\delta(n'),$$

where the bar denotes an ensemble average, the asterisk the complex conjugate, $\delta(k_1')$, etc., the Dirac delta functions and

$$\Pi(y,\mathbf{k},n) = (2\pi)^{-3}\iint \overline{\zeta(y,y_1,y_2,\tau)\,\zeta(y,y_1+r_1,y_2+r_2,\tau+\tau')} \times \exp\{-i(\mathbf{k}.\mathbf{r}+n\tau')\}\,d\mathbf{r}\,d\tau'$$

the wave-number-frequency spectrum of the pressure function. Thus

$$\Pi(y,\mathbf{k},n) = \iint \overline{\varpi(y,\mathbf{k},n)\,\varpi^*(y,\mathbf{k}-\mathbf{k}',n-n')}\,d\mathbf{k}'\,dn', \tag{5.12}$$

the integral being over a small range containing $k_1' = k_2' = n' = 0$. Similarly,

$$\begin{aligned}\Psi(Y,\mathbf{k},n) &= \iint \overline{Z(Y,\mathbf{k},n)\,Z^*(Y,\mathbf{k}-\mathbf{k}',n-n')}\,d\mathbf{k}'\,dn' \\ &= (2\pi)^{-3}\iint \overline{v_3(Y,y_1,y_2,\tau)\,v_3(Y,y_1+r_1,y_2+r_2,\tau+\tau')} \\ &\qquad\qquad \times \exp\{-i(\mathbf{k}.\mathbf{r}+n\tau')\}\,d\mathbf{r}\,d\tau',\end{aligned} \tag{5.13}$$

for the spectrum of the v_3-velocity fluctuations.

For definiteness, we will consider the region $y > 1$ above the shear zone, and constructing $\Pi(y,\mathbf{k},n)$ from these last two expressions, (5.10) and (5.7), it is found that

$$\Pi(\mathbf{k},n) \sim \tfrac{1}{2}[(-\tfrac{3}{4})!]^2\,\frac{\gamma^2 M^{\frac{3}{2}}\Omega^{\frac{1}{2}}(Y)}{A^2(Y)}\,\frac{k_1^2\Psi(Y,\mathbf{k},n)}{|k_1+n|}. \tag{5.14}$$

The dependence on y of the right-hand side has disappeared, reflecting the fact that, in the present theory, the statistical properties of the pressure fluctuations in the free stream are independent of the distance from the shear zone of the plane in which they are observed. This theory has neglected the self-convection of the sound waves which leads to the development of shock waves as y increases if $\overline{\zeta^2}$ is sufficiently large, and as we have stated, can be expected to be accurate only near the shear zone before this process has distorted the pressure waves appreciably.

The instantaneous wave-number spectrum is obtained by integrating (5.14) over all frequencies n:

$$\begin{aligned}\Pi(\mathbf{k}) &= \int_{-\infty}^{\infty}\Pi(\mathbf{k},n)\,dn \\ &= (2\pi)^{-2}\int \overline{\zeta(y,y_1,y_2,\tau)\,\zeta(y,y_1+r_1,y_2+r_2,\tau)}\,e^{-i\mathbf{k}.\mathbf{r}}\,d\mathbf{r}.\end{aligned} \tag{5.15}$$

Since $n+k_1V(Y) = 0$,

$$dn = -k_1\Omega(Y)\,dY, \tag{5.16}$$

and integration over all frequencies n with fixed $\mathbf{k}$ corresponds to integration over the position Y of the critical layer across the shear zone. Thus

$$\Pi(\mathbf{k}) \sim \tfrac{1}{2}[(-\tfrac{3}{4})!]^2\gamma^2 M^{\frac{3}{2}}k_1^2\int_{-\infty}^{\infty}\frac{\Omega^{\frac{3}{2}}(Y)\,\Psi(Y,\mathbf{k},n)}{A^2(Y)\,[1-V(Y)]}\,dY, \tag{5.17}$$

where $n = -k_1V(Y)$.

It should be pointed out that this expression is likely to provide a slight numerical overestimate of the pressure spectrum, since the integral includes layers near the outside edges of the shear zone, where the approximation (5.10) is not appropriate and, indeed, where the eddy wave mechanism may not be operative, since the difference between the free-stream velocity and the eddy convection velocity is subsonic. However, if M is large, these regions constitute only a small part of the total shear zone, and the error in integrating (5.14) over the whole zone instead of over the supersonic region only should be relatively small. The relative contributions to $\Pi(\mathbf{k})$ from critical layers symmetrically spaced above and below the centre-line $y = 0$ is demonstrated clearly in this expression, since $V(Y)$ is positive in the former case and negative in the latter ($|V| \leqslant 1$). We might anticipate that a possible divergence of the integral at the upper limit, where $1 - V(Y) \to 0$, is avoided by the fact that $\Omega(Y) \to 0$ also.

The integral in (5.17) can be simplified further by making use of a plausible approximation on the nature of the turbulence in the shear zone. It will be supposed that the spectrum of the v_3-velocity covariance with time delay is independent of Y provided the observations are made in a frame of reference moving with the local mean velocity. It is shown below that this statement really involves two approximations, the first being that the mean square value of these velocity fluctuations is independent of the position in the shear layer $|y| < 1$ at which they are measured. Our experience with subsonic free turbulent flows suggests that this is a good approximation for the fully turbulent regions; Townsend (1956) quotes a considerable body of experimental support. It is difficult to see why this general property should not also be true in the supersonic shear zone. The second requirement is that the integral time scale of the velocity fluctuations, *measured in a frame of reference moving with the local convective mean velocity*, is also independent of Y. This also is strongly suggested by the observed near-homogeneity of such flows for $M < 1$.

It can be shown simply (Phillips 1957, §4.1) that if $\Psi(Y, \mathbf{k}, \tau)$ represents the two-dimensional wave-number spectrum with time delay τ measured in a frame of reference at rest, then the corresponding quantity observed in a frame of reference moving with velocity $\mathbf{V}$ is

$$e^{i(\mathbf{k}\cdot\mathbf{V})\tau}\,\Psi(Y, \mathbf{k}, \tau), \tag{5.18}$$

and this quantity we suppose to be independent of Y. Our wave-number-frequency spectra of this problem are all measured in a frame at rest, and are given by

$$\begin{aligned}\Psi(Y, \mathbf{k}, n) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\Psi(Y, \mathbf{k}, \tau)\, e^{-in\tau}\, d\tau \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty}\Psi(Y, \mathbf{k}, \tau)\exp\{ik_1 V(Y)\tau\}\, d\tau,\end{aligned} \tag{5.19}$$

since $n = -k_1 V(Y)$. In our problem, $\mathbf{V} = (V, 0, 0)$, so that, under our assumption, the integrand is independent of Y and equal to, in particular, its value at $Y = 0$. Thus

$$\begin{aligned}\Psi(Y, \mathbf{k}, n) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\Psi(0, \mathbf{k}, \tau)\, d\tau \\ &= \pi^{-1}\Psi(0, \mathbf{k})\,\theta(\mathbf{k}), \quad \text{say},\end{aligned} \tag{5.20}$$

where $\Psi(0, \mathbf{k})$ represents the usual two-dimensional wave-number spectrum with zero time delay at $Y = 0$ and $\theta(\mathbf{k})$ the integral time scale of the spectrum of the v_3-fluctuations measured at the centre-line where $V(Y) = 0$. If, in addition the mean density is approximately constant across the shear zone, the variation in $A(Y)$ can be neglected. We might note in passing the inference that if the shear zone is hot, so that $A(Y)$ is significantly less than unity, then the pressure radiation outside the shear zone will be increased by a factor proportional to the inverse square of the sonic speed ratio.

With these simplifications, (5.17) becomes

$$\Pi(\mathbf{k}) \sim (2\pi)^{-1}[(-\tfrac{3}{4})!]^2 \gamma^2 M^{\frac{3}{2}} k_1^2 \Psi(0,\mathbf{k})\,\theta(\mathbf{k}) \int_{-\infty}^{\infty} \frac{\Omega^{\frac{3}{2}}(Y)\,dY}{1-V(Y)}. \qquad (5.21)$$

The integrand in this last expression gives the relative contributions to $\Pi(\mathbf{k})$ (or to $\overline{\zeta^2}$) from the various layers Y within the shear zone. Clearly the greater contributions to the disturbance above the layer originate from the region $Y > 0$ where $V(Y)$ is positive. The magnitude of the integral depends only on the mean velocity profile in the shear zone, and is fairly insensitive to the details of its shape. $V(Y)$ increases from approximately -1 to 1 as Y increases from -1 to 1. If we approximate $V(Y)$ by

$$|V(Y)| = 1 - \exp(-2|Y|),$$

the integral has the value 3·08 approximately of which the region $0 < Y < \infty$ contributes about 90 %. If we take $V(Y)$ as the error function erfc $(2Y)$, then the integral is approximately 5·13, of which 82 % is contributed by the upper half of the shear zone. It seems likely then, that the value of the integral will, under ordinary circumstances, be about 4, the probable error lying well within a factor of two. Thus

$$\Pi(\mathbf{k}) \sim \frac{2}{\pi}[(-\tfrac{3}{4})!]^2 \gamma^2 M^{\frac{3}{2}} k_1^2 \Psi(\mathbf{k})\,\theta(\mathbf{k}), \qquad (5.22)$$

where $\Psi(\mathbf{k}) \equiv \Psi(0,\mathbf{k})$.

Equation (5.22) expresses the basic result of this part of the paper. It emphasizes the dependence of the pressure spectrum on the spectrum of $(\partial v_3/\partial y_1)$, which is in accord with our conception of the mechanism of generation in terms of Mach waves arising from the Fourier components of the fluctuations in velocity normal to the shear zone. The time scale $\theta(\mathbf{k})$ enters the problem because of the statistical nature of the process; the greater the time scale $\theta(\mathbf{k})$ (or the longer the eddy lifetime) the greater is the coherence in the contributions to the pressure field and the sum of all contributions is increased.

If the fluctuations in pressure outside the shear zone are not a large fraction of the mean pressure at infinity, p_0, then the spectrum (5.22) can be interpreted directly as the spectrum of the pressure fluctuation level. For

$$\zeta = A(y)\log\left(\frac{p}{p_0}\right) \simeq \frac{p-p_0}{p_0}$$

outside the shear layer, where $A(y) = 1$, if $(p-p_0) \ll p_0$. This requirement is equivalent to restricting our considerations to situations in which only weak shocks are generated by the turbulence, and this is consistent with our earlier approximations. At extremely high Mach numbers, $p-p_0$ may be comparable in root mean square to p_0, but the whole theory may become unreliable, since dissociation and other phenomena may become important.

To form an estimate of $\overline{(p-p_0)^2}$ outside the shear zone, some information is needed concerning $\theta(\mathbf{k})$. If the lifetime of the eddies is determined by the straining process, as is quite conceivable in supersonic shear flow where the turbulence level is considerably smaller than in a corresponding subsonic flow, the characteristic (dimensional) time is of order $(|\partial U/\partial x_3|_{\max})^{-1}$ (Townsend 1956, p. 96), so that the dimensionless $\theta(\mathbf{k})$ is of order unity. On the other hand, if the process determining the life-time is the turbulent rate of strain due to the energy containing eddies, or some more complicated process involving compressibility effects, the time scale may be different, and in view of our lack of information in supersonic turbulence, it is difficult at this stage to evaluate the possible alternatives. It seems that to take $\theta(\mathbf{k}) = 1$ is probably the best course open; it is certainly the simplest.

Equation (5.22) can then be integrated over all wave-numbers $\mathbf{k}$ (including those outside the fan of figure 2: the spurious contribution from these will be small because of the k_1^2 factor in (5.22)) and we obtain

$$\frac{\overline{(p-p_0)^2}}{p_0^2} \sim \frac{2}{\pi}[(-\tfrac{3}{4})!]^2\gamma^2 M^{\frac{3}{2}}\overline{\left(\frac{\partial v_3}{\partial y_1}\right)^2}$$

$$\simeq 8\gamma^2 M^{\frac{3}{2}}\overline{\left(\frac{\partial v_3}{\partial y_1}\right)^2} \tag{5.23}$$

in the notation of (3.5). Restoring the dimensional quantities of § 3,

$$\frac{\overline{(p-p_0)^2}}{p_0^2} \sim 8\gamma^2 U^{\frac{3}{2}} a_0^{-\frac{3}{2}}\overline{\left(\frac{\partial u_3}{\partial x_1}\right)^2}\frac{L^2}{U^2}, \tag{5.24}$$

$$= 8\gamma^2 U^{\frac{3}{2}} a_0^{-\frac{3}{2}}\left(\frac{\overline{u_3^2}}{U^2}\right)\frac{L^2}{l^2}, \tag{5.25}$$

where l is the differential length scale in the streaming direction of the velocity fluctuations normal to the shear zone.†

It is interesting to contrast the behaviour predicted by (5.25) for large M with the U^8 law found by Lighthill when $M \ll 1$. If $\overline{u_3^2}/U^2$ is only weakly dependent on U when the Mach number is large,‡ the mean-square-pressure fluctuations increase only as the $\frac{3}{2}$ power of the velocity for $M \gg 1$. A decrease in the steepness of the curve $\overline{(p-p_0)^2}/p_0^2$ *vs* U is very evident from the little experimental data that is available, of which perhaps the best is provided by the recent measurements of Laufer (1959). These were made inside a supersonic wind-tunnel and the major source of the sound field appeared to be the turbulent boundary layers along the walls. Laufer found that the over-all variation of $\overline{(p-p_0)^2}$ was as M^4, approximately, but that the boundary-layer thickness also increased with Mach number, a little faster than linearly. The radiation from a supersonic turbulent

† The result (5.24) is applicable also (possibly with a modified $\theta(\mathbf{k})$) to the pressure fluctuations generated by the instability of a supersonic laminar shear zone.

‡ It seems likely that $\overline{u_3^2}/U^2$ will vary only slowly with M, if at all, in view of the remarks in § 6 concerning radiation damping.

boundary layer could be considered by the methods of this paper, replacing the radiation condition at $y = -\infty$ by the appropriate wall boundary condition. It is very likely that the result would be similar to (5.25) except for a difference in the numerical constant since the basic mechanism of the sound generation is the same. This suggests that, for the boundary layer also

$$\frac{\overline{(p-p_0)^2}}{p_0^2} \propto M^{\frac{3}{2}} \frac{\delta^2}{l^2} \frac{\overline{u_3^2}}{U^2}.$$

where δ is the boundary-layer thickness, and if l and $\overline{u_3^2}/U^2$ are approximately independent of Mach number (see footnote on previous page), then

$$\frac{\overline{(p-p_0)^2}}{p_0^2 M^{\frac{3}{2}} \delta^2} = \text{constant.} \tag{5.26}$$

Laufer's data are plotted in figure 3 and, although there is some scatter, there is no evident trend with Mach number of the quantity on the left-hand side of (5.26). It therefore appears that Laufer's M^4 variation represents the combined

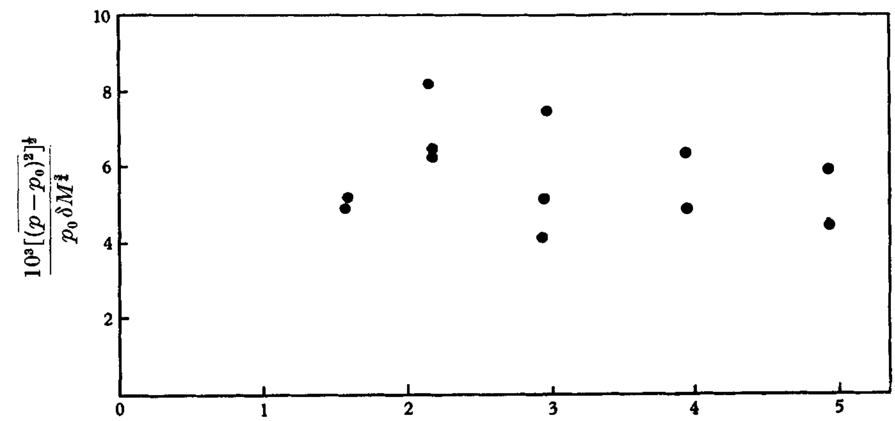

FIGURE 3. Pressure fluctuations from a supersonic turbulent boundary layer (after Laufer 1959).

effects of the $M^{\frac{3}{2}}$ factor given by the present theory and a variation of the boundary-layer thickness δ, as $M^{\frac{5}{4}}$ approximately. The present theory also indicates that the pressure field external to the shear flow is being swept past a point fixed in the external fluid with speed of order U. In the boundary layer, this convection speed would be lower, of order $\frac{1}{2}U$ corresponding to the speed of the flow outside the viscous and buffer layers. This also is in rough agreement with Laufer's results. It appears that these measurements give promising, but tentative support to the theory, though more measurements are required, both on the sound field and on the boundary-layer structure before firm conclusions can be drawn.

6. The directional distribution

The directional distribution of the radiation above the shear zone is found most readily from (5.21). Relative to the fluid in $y > 1$, the critical layer at Y is being convected with speed $U[1-V(Y)]$, and from simple geometrical considerations, if $U[1-V(Y)] > a_0$, the direction of propagation relative to the outside stream of waves originating at the layer Y is given by

$$\begin{aligned} \alpha &= \cos^{-1}\frac{a_0}{U[1-V(Y)]} \\ &= \cos^{-1}\{M[1-V(Y)]\}^{-1}. \end{aligned} \tag{6.1}$$

The relative contributions to $\overline{(p-p_0)^2}$ from the layers between Y, $Y+dY$ are given by (5.21) as

$$\frac{\Omega^{\frac{3}{2}}dY}{1-V(Y)}, \tag{6.2}$$

and each layer Y is associated with an angle of propagation α specified by (6.1). By differentiation of (6.1)

$$\Omega(Y)\,dY = -\frac{\sin\alpha}{M\cos^2\alpha}\,d\alpha, \tag{6.3}$$

so that the directional distribution function is given by (6.1), (6.2) and (6.3) as

$$f(\alpha) = \Omega^{\frac{1}{2}}(Y)\tan\alpha, \tag{6.4}$$

where $Y = Y(\alpha)$ is given by (6.1). The approximate shape of the directional distribution can be found by assuming a shape of the velocity profile, a convenient form being given by

$$|V(Y)| = 1-e^{-2|Y|},$$

and the directional distribution function $f(\alpha)$ is illustrated in figure 4 for this profile for Mach numbers of 2, 4 and 6 corresponding to Mach numbers based on the velocity *difference* of 4, 8 and 12. As the Mach number increases, the maximum of the directional distribution moves towards 90° and becomes sharper, since most of the Mach lines from the shear zone are inclined at quite a small angle to the direction of the free stream.

The cut-off in figure 4, which occurs when $\cos\alpha = (2M)^{-1}$ will be modified in practice by the presence of scattering phenomena introduced by the turbulence, and by pressure wave interactions, both of which have been neglected in the present theory, and which would result in the radiation of some sound in the backwards direction, at angles greater than 90°.

Figure 4 indicates that the direction of radiation observed at rest relative to the mean stream above the shear layer is for large M concentrated in directions near but less than 90°. This property makes possible a simple estimate of the acoustic energy flux N per unit area from the shear zone, using expressions derived by Ribner (1958). If $\alpha_{\max}$ is the direction in which most of the energy travels

$$\begin{aligned} N = \frac{\overline{(p-p_0)^2}}{\rho_0 a_0}\sin\alpha_{\max} &\simeq \frac{\overline{(p-p_0)^2}}{\rho_0 a_0} \\ &\sim 8\rho_0 U^{\frac{3}{2}}\alpha_0^{\frac{3}{2}}\left(\frac{\overline{u_3^2}}{U^2}\right)\frac{L^2}{l^2}, \end{aligned} \tag{6.5}$$

from (5.15), and the acoustic efficiency η is

$$\eta = \frac{\text{acoustic energy flux}}{\rho_0 U^3}$$

$$\propto M^{-\frac{3}{2}} \frac{\overline{u_3^2}}{U^2} \frac{L^2}{l^2}. \tag{6.6}$$

This decreasing acoustic efficiency with increasing M for large Mach number contrasts with an increase as M^5 when M is small shown by Lighthill (1952). This implies that at some Mach number of order one or two the acoustic efficiency

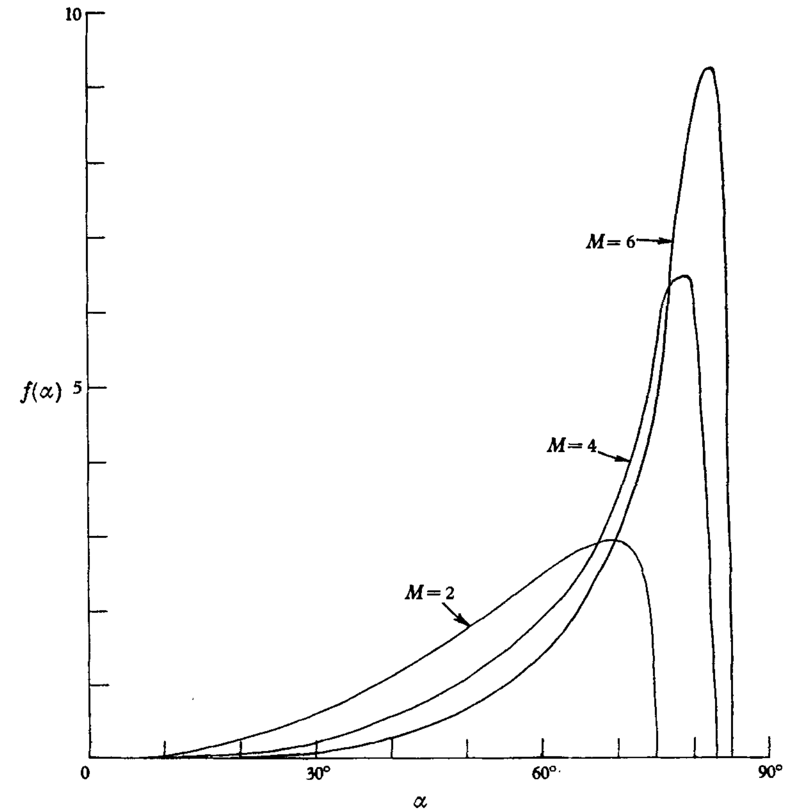

FIGURE 4. The directional distribution function for $M = 2$, 4 and 6, corresponding to Mach numbers for the velocity difference across the shear zone of 4, 8 and 12, respectively.

is a maximum, decreasing rapidly as $M \to 0$ and more slowly as $M \to \infty$. It also implies that the loss of energy from the turbulence by acoustic radiation is always small compared with the energy supply from the mean flow, and the effects of radiation damping are likely to be small.

These inferences are supported by the little data that is available. Lilley (1958), in discussing the measurements of Lassiter & Heitkotter (1954), suggests that a qualitative behaviour of this kind is indicated since the measured acoustic efficiency of a round jet at $M = 5$ is approximately the same as found at $M = 1{\cdot}8$

by Sanders & Callaghan (1956) below which it drops off rapidly. Lassiter & Heitkotter also measured the directional distribution of the radiation from a round jet whose exit Mach number was 3·16. Since most of their sound originated in the shear layer near the exit, one might hope to compare their measurements with the results of this paper. Their directional maximum occurred at about 50°, compared with the 57° predicted by this theory for the corresponding Mach number. These angles are significantly larger than those found for the maximum intensity in subsonic jets (about 30°). As α increased beyond 60°, the observed intensity dropped rapidly to a level about 60 db below the maximum, in good qualitative accord with the shapes of the curves shown in figure 4.

Although these preliminary indications are again quite promising, a great deal more data will have to be obtained before more detailed comparisons are possible. There still remains the question of the interaction of the random pattern of weak shock waves outside the shear layer, but that is another story.

I am greatly indebted to Dr L. S. G. Kovasznay, Dr M. Morkovin and Dr S. Corrsin for the stimulation of discussing this work with them. It was supported by the Office of Naval Research under Contract Nonr 248 (38).

Appendix. Asymptotic solution for large M

Equation (3.8) can be expressed as

$$\frac{d^2\varpi(y)}{dy^2}+\{M^2q^2(y)-k^2\}\varpi(y) = -\frac{M^2}{A^2(y)}\Gamma(y), \tag{A.1}$$

where $k^2 \gg |A''/A|$ and

$$q(y) = \pm\frac{n+k_1 V(y)}{A(y)}, \tag{A.2}$$

the positive or negative sign being taken according as k_1 is positive or negative. If $|k_1| > |n|$, $q(y)$ has a single zero at $y = Y$, say, and $q(y) > 0$ for $y > Y$ and $q(y) < 0$ for $y < Y$. Let

$$\xi = \chi(y), \quad \eta = \varpi(y)\,\psi(y), \tag{A.3}$$

and on substitution into (A.1), we find

$$\frac{d^2\eta}{d\xi^2}+\frac{1}{\chi'}\left\{\frac{\chi''}{\chi'}-\frac{2\psi'}{\psi}\right\}\frac{d\eta}{d\xi}+\left\{\frac{M^2q^2-k^2}{(\chi')^2}+\frac{\psi}{(\chi')^2}\frac{d^2}{dy^2}\psi^{-1}\right\}\eta = -\frac{\psi}{(\chi')^2}\frac{M^2}{A^2(y)}\Gamma(y). \tag{A.4}$$

Now choose ψ so that

$$\frac{\chi''}{\chi'} = \frac{2\chi'}{\psi},$$

or

$$\psi = (\chi')^{\frac{1}{2}}, \tag{A.5}$$

and determine χ so that

$$\frac{q^2(y)}{(\chi')^2} = \chi^2 = \xi^2. \tag{A.6}$$

Thus

$$\xi = \chi(y) = \left\{2\int_Y^y q(y)\,dy\right\}^{\frac{1}{2}}. \tag{A.7}$$

The positive values of the fractional indices are to be taken and the lower limit of the integral is chosen as Y so that ξ is real for all y. Clearly $0 \leqslant \xi < \infty$ and the transformation $y = y(\xi)$ is double valued with branch I corresponding to $Y \leqslant y < \infty$ and branch II to $-\infty < y \leqslant Y$. From (A. 5) and (A. 7)

$$\eta = \varpi(y)\,\psi(y) = q^{\frac{1}{2}}\{2\textstyle\int q\,dy\}^{-\frac{1}{4}}\,\varpi(y), \tag{A. 8}$$

where the limits of integration are understood to be as specified above. With the transformation (A. 7) and (A. 8), (A. 4) becomes

$$\frac{d^2\eta}{d\xi^2} + M^2\xi^2\eta = \left\{k^2 + \frac{\chi'''}{2\chi'} - \frac{3}{4}\left[\frac{\chi''}{\chi'}\right]^2\right\}\frac{\eta}{(\chi')^2} - \frac{M^2\Gamma(y)}{(\chi')^{\frac{3}{2}}A^2(y)}. \tag{A. 9}$$

Now

$$\frac{d\xi}{dy} = \chi' = \frac{q(y)}{\xi}, \tag{A. 10}$$

and $|q(y)|$ is very nearly constant outside the shear zone, i.e. for $|y| > 1$. It follows by differentiation of (A. 10) that

$$\left\{k^2 + \frac{\chi'''}{2\chi'} - \frac{3}{4}\left[\frac{\chi''}{\chi'}\right]^2\right\}(\chi')^{-2} = \frac{k^2\xi^2}{Q^2} + O(\xi^{-2}), \tag{A. 11}$$

as $\xi \to \infty$, where

$$\left.\begin{aligned} Q = Q_+ &= \lim_{y\to\infty} q(y) \quad \text{for branch I,} \\ Q = Q_- &= \lim_{y\to-\infty} q(y) \quad \text{for branch II.}\end{aligned}\right\} \tag{A. 12}$$

From (A. 2), clearly

$$\left.\begin{aligned} Q_+ &= n + k_1, \\ Q_- &= \frac{n-k_1}{A(-\infty)},\end{aligned}\right\} \tag{A. 13}$$

since $A(\infty) = V(\infty) = 1$, $V(-\infty) = -1$. The behaviour (A. 11) suggests that we incorporate the term $k^2\xi^2/Q^2$ into the left-hand side of (A. 9), giving

$$\frac{d^2\eta}{d\xi^2} + \mu^2\xi^2\eta = g(\xi)\,\eta + h(\xi), \tag{A. 14}$$

where, for branch I,

$$\mu = \left\{M^2 - \frac{k^2}{Q_+^2}\right\}^{\frac{1}{2}}, \tag{A. 15}$$

$$\left.\begin{aligned} g(\xi) &= \left\{k^2 + \frac{\chi'''}{2\chi'} - \frac{3}{4}\left[\frac{\chi''}{\chi'}\right]^2\right\}(\chi')^{-2} - \frac{k^2\xi^2}{Q_+^2}, \\ h(\xi) &= -\frac{M^2\Gamma(y)}{(\chi')^{\frac{3}{2}}A^2(y)}.\end{aligned}\right\} \tag{A. 16}$$

From (A. 10), $(\chi')^{-1}$ is bounded near $\xi = 0$, and from (A. 11) $g(\xi)$ is $O(\xi^{-2})$ as $\xi \to \infty$, so that $g(\xi)$ is integrable over $(0, \infty)$. Furthermore, $\Gamma(y) = 0$ outside the shear zone, so that $h(\xi)$ also is integrable over $(0, \infty)$. The governing equation for branch II $(-\infty < y \leqslant Y)$ is the same as (A. 14) except that Q_+ is replaced by Q_-. We seek an asymptotic solution to (A. 14) which is to be matched to the corresponding solution for branch II at $\xi = 0$.

The following theorem will be used. If $K(\xi, t)$ is the solution to

$$\frac{d^2K}{d\xi^2} + \mu^2\xi^2 K = 0 \tag{A.17}$$

satisfying the conditions

$$K(\xi, t) = 0, \quad \frac{d}{d\xi}K(\xi, t) = 1 \quad \text{when} \quad \xi = t \tag{A.18}$$

and $H(\xi)$ is *any* solution to (A. 17), then the Volterra integral equation

$$\eta(\xi) = H(\xi) + \int_0^\xi K(\xi, t)\,[g(t)\,\eta(t) + h(t)]\,dt \tag{A.19}$$

satisfies (A. 14). The proof is given by Erdélyi (1956, p. 99). Note that $\eta(\xi)$ and $H(\xi)$ have the same value and the same derivative at $\xi = 0$.

The fundamental solutions to (A. 17) are

$$(\mu\xi^2)^{\frac{1}{2}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2), \quad (\mu\xi^2)^{\frac{1}{2}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2),$$

and the kernel K satisfying the conditions (A. 18) is

$$K(\xi, t) = \frac{\pi}{2\sqrt{2}\,\mu^{\frac{1}{2}}}\{(\mu\xi^2)^{\frac{1}{2}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\,(\mu t^2)^{\frac{1}{2}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu t^2) - (\mu\xi^2)^{\frac{1}{2}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\,(\mu t^2)^{\frac{1}{2}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu t^2)\}. \tag{A.20}$$

To find the asymptotic solution to (A. 19) let

$$\eta_0(\xi) = H(\xi) + \int_0^\xi K(\xi, t)\,h(t)\,dt. \tag{A.21}$$

If

$$\left.\begin{aligned} \Phi_1(\tfrac{1}{2}\mu\xi^2) &= \int_{\frac{1}{2}\mu\xi^2}^\infty u^{-\frac{1}{4}}J_{-\frac{1}{4}}(u)\,du, \\ \Phi_2(\tfrac{1}{2}\mu\xi^2) &= \int_{\frac{1}{2}\mu\xi^2}^\infty u^{-\frac{1}{4}}J_{\frac{1}{4}}(u)\,du, \end{aligned}\right\} \tag{A.22}$$

then

$$(\mu t^2)^{\frac{1}{2}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu t^2) = -2^{-\frac{1}{4}}\mu^{-\frac{1}{2}}\frac{d}{dt}\,\Phi_1(\tfrac{1}{2}\mu t^2),$$

$$(\mu t^2)^{\frac{1}{2}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu t^2) = -2^{-\frac{1}{4}}\mu^{-\frac{1}{2}}\frac{d}{dt}\,\Phi_2(\tfrac{1}{2}\mu t^2),$$

so that substituting (A. 20) into (A. 21), we have

$$\begin{aligned} \eta_0(\xi) &= H(\xi) - \frac{\pi}{2\sqrt{2}\,\mu}\left\{(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\int_0^\xi h(t)\frac{d}{dt}\,\Phi_1(\tfrac{1}{2}\mu t^2)\,dt \right.\\ &\qquad \left. - (\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\int_0^\xi h(t)\frac{d}{dt}\,\Phi_2(\tfrac{1}{2}\mu t^2)\,dt\right\}, \\ &= H(\xi) + \frac{\pi}{2\sqrt{2}\,\mu}\{(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\,[h(+0)\,\Phi_1(0) - h(\xi)\,\Phi_1(\tfrac{1}{2}\mu\xi^2)] \\ &\qquad - (\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\,[h(+0)\,\Phi_2(0) - h(\xi)\,\Phi_2(\tfrac{1}{2}\mu\xi^2)] + O(\mu^{-\frac{1}{2}})\}, \end{aligned} \tag{A.23}$$

on integration by parts, where

$$h(+0) = \lim_{y\to Y+0}\left\{-\frac{M^2\Gamma(y)}{(\chi')^{\frac{3}{2}}\,A^2(y)}\right\}.$$

The next approximation, $\eta_0+\eta_1$, is obtained by substituting η_0 into the integral of (A. 19) (Erdélyi 1956). Equation (A. 23) is of the form

$$\eta_0(\xi) = H(\xi)+\mu^{-1}G(\xi,\mu),$$

so that

$$\int_0^\xi K(\xi,t)\,\eta_0(t)\,g(t)\,dt = \int_0^\xi K(\xi,t)\,H(t)\,g(t)\,dt+\mu^{-1}\int_0^\xi K(\xi,t)\,G(t,\mu)\,g(t)\,dt,$$

and integration by parts again shows that

$$\left|\int_0^\xi K(\xi,t)\,\eta_0(t)\,g(t)\,dt\right| \leqslant \mu^{-1}H(0)\,\Lambda,$$

where Λ is a finite constant. Thus

$$\eta_1(\xi) = H(\xi)\left[1+\mu^{-1}\frac{H(0)\,\Lambda}{H(\xi)}\right]+O(\mu^{-1}). \tag{A. 24}$$

If only the leading term in the asymptotic expansion is required, it is clear that the second term in the square bracket is negligible except near the zeros of $H(\xi)$. However, if we can take $H(\xi)$ as

$$H(\xi) = A(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)+B(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2), \tag{A. 25}$$

where the ratio of the constants A/B is complex, then $H(\xi)$ has no zeros and (A. 23) is a uniformly valid asymptotic solution as $\mu\to\infty$. It will be shown below that such constants can be found for which the radiation and matching conditions in the present problem are satisfied.

Outside the shear zone, $h(\xi) = 0$, and

$$\begin{aligned}\eta(\xi) &= (\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\left[A+\frac{\pi}{2\sqrt{2}\,\mu}h(+0)\,\Phi_1(0)\right]\\ &\qquad +(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)\left[B-\frac{\pi}{2\sqrt{2}\,\mu}h(+0)\,\Phi_2(0)\right]\\ &= \alpha(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2)+\beta(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,J_{-\frac{1}{4}}(\tfrac{1}{2}\mu\xi^2),\quad \text{say,}\end{aligned} \tag{A. 26}$$

where

$$\left.\begin{aligned}\alpha &= A+\frac{\pi}{2\sqrt{2}\,\mu}h(+0)\,\Phi_1(0),\\ \beta &= B-\frac{\pi}{2\sqrt{2}\,\mu}h(+0)\,\Phi_2(0).\end{aligned}\right\} \tag{A. 27}$$

From (A. 8),

$$\varpi(y) = q^{-\frac{1}{2}}\{2\textstyle\int q\,dy\}^{\frac{1}{2}}\,\eta = 2^{\frac{1}{4}}\mu^{-\frac{1}{4}}q^{-\frac{1}{2}}(\tfrac{1}{2}\mu\xi^2)^{\frac{1}{4}}\,\eta, \tag{A. 28}$$

so that outside the shear zone, as $\mu\to\infty$,

$$\varpi(y) \sim 2^{\frac{3}{4}}\pi^{-\frac{1}{2}}\mu^{-\frac{1}{4}}Q_{\mp}^{-\frac{1}{2}}[\alpha\cos(\tfrac{1}{2}\mu\xi^2-\tfrac{3}{8}\pi)+\beta\cos(\tfrac{1}{2}\mu\xi^2-\tfrac{1}{8}\pi)], \tag{A. 29}$$

where we have used the asymptotic relations

$$\left.\begin{aligned}x^{\frac{1}{2}}J_{\frac{1}{4}}(x) &\sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos(x-\tfrac{3}{8}\pi),\\ x^{\frac{1}{2}}J_{-\frac{1}{4}}(x) &\sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos(x-\tfrac{1}{8}\pi),\end{aligned}\right\} \tag{A. 30}$$

as $x \to \infty$. Equation (A. 29) can be written alternatively as

$$\varpi(y) \sim (8\pi^2\mu Q_+^2)^{-\frac{1}{4}}\{[\sqrt{2}\,\alpha+(1+i)\,\beta]\exp i(\mu\int q\,dy-\tfrac{3}{8}\pi) + [\sqrt{2}\,\alpha+(1-i)\,\beta]\exp -i(\mu\int q\,dy-\tfrac{3}{8}\pi)\}. \tag{A. 31}$$

The solution and its asymptotic form for branch II corresponding to $-\infty < y \leqslant Y$ can be expressed similarly.

$$\eta(\xi) = (\tfrac{1}{2}\lambda\xi^2)^{\frac{1}{2}}J_{\frac{1}{4}}(\tfrac{1}{2}\lambda\xi^2)\left\{C+\frac{\pi}{2\sqrt{2}\,\lambda}[h(-0)\,\Phi_1(0)-h(\xi)\,\Phi_1(\tfrac{1}{2}\lambda\xi^2)]\right\} + (\tfrac{1}{2}\lambda\xi^2)^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}\lambda\xi^2)\left\{D-\frac{\pi}{2\sqrt{2}\,\lambda}[h(-0)\,\Phi_2(0)-h(\xi)\,\Phi_2(\tfrac{1}{2}\lambda\xi^2)]\right\}+O(\lambda^{-\frac{3}{2}}), \tag{A. 32}$$

where

$$\left.\begin{aligned} \lambda &= \left\{M^2-\frac{k^2}{Q_-^2}\right\}^{\frac{1}{2}}, \\ h(-0) &= \lim_{y\to Y-0}\left\{\frac{-M^2\Gamma(y)}{(\chi')^{\frac{3}{2}}A^2(y)}\right\}, \\ &= -ih(+0), \end{aligned}\right\} \tag{A. 33}$$

since

$$\begin{aligned} \chi' &= q(y)/\xi \quad \text{from (A. 10)} \\ &= \frac{\Omega(y-Y)}{\Omega^{\frac{1}{2}}\,|y-Y|} \quad \text{near} \quad y = Y, \end{aligned}$$

$$\left.\begin{aligned} &\to \Omega^{\frac{1}{2}} \quad \text{as} \quad y\to Y+0, \\ &\to -\Omega^{\frac{1}{2}} \quad \text{as} \quad y\to Y-0, \end{aligned}\right\} \tag{A. 34}$$

and C and D are further constants. Outside the shear zone, as $\lambda \to \infty$,

$$\varpi(y) \sim (8\pi^2\lambda Q_-^2)^{-\frac{1}{4}}\{[\sqrt{2}\,\gamma+(1+i)\,\delta]\exp i(\lambda\int q\,dy-\tfrac{3}{8}\pi) + [\sqrt{2}\,\gamma+(1-i)\,\delta]\exp -i(\lambda\int q\,dy-\tfrac{3}{8}\pi)\}, \tag{A. 35}$$

where

$$\left.\begin{aligned} \gamma &= C-\frac{i\pi}{2\sqrt{2}\,\lambda}h(+0)\,\Phi_1(0), \\ \delta &= D+\frac{i\pi}{2\sqrt{2}\,\lambda}h(+0)\,\Phi_2(0). \end{aligned}\right\} \tag{A. 36}$$

Determination of the constants

We seek to determine the constants A, B, C, D by applying radiation conditions outside the shear zone and matching conditions between the two branches at $\xi = 0$. If W represents the component of the velocity of the convected wave outside the shear zone, observed in the direction of y increasing, then

$$\frac{\partial\zeta}{\partial t}+W\frac{\partial\zeta}{\partial y} = 0,$$

or in terms of our Fourier transforms

$$in\varpi(y)+W\frac{\partial\varpi(y)}{\partial y} = 0.$$

Thus

$$in\varpi(y)\bigg/\left(\frac{\partial\varpi(y)}{\partial y}\right) = -W.$$

The two parts of the solution (A. 31) are of the form $\exp(\pm i\mu \int q\,dy)$; let us consider the particular wave motion represented by

$$\varpi(y) \propto \exp(-i\mu \int q\,dy). \tag{A.37}$$

For such waves

$$W = -in\varpi \Big/ \frac{\partial \varpi}{\partial y} = \frac{n}{\mu q(y)},$$

$$= \frac{n}{\mu(n+k_1)}, \tag{A.38}$$

for $y > 1$, from (A. 13). But from (4.3), $n = -k_1 V_c$ approximately, where V_c is the convection velocity at the critical layer Y, so that from (A. 42)

$$W = \{\mu(1 - V_c^{-1})\}^{-1}. \tag{A.39}$$

Since $|V_c| \leqslant 1$, W is positive or negative according as $V_c < 0$ or $V_c > 0$.

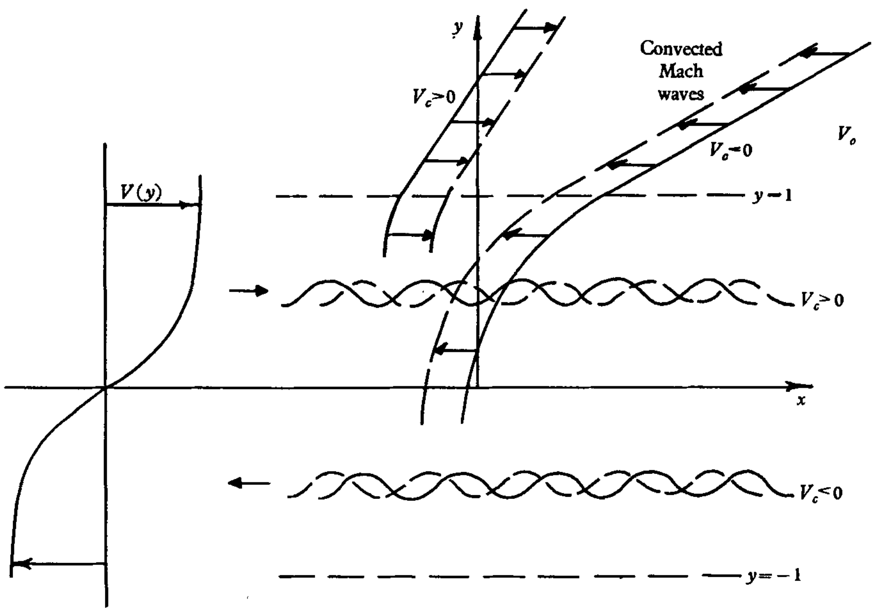

FIGURE 5. The convection of eddy Mach waves.

Figure 5 shows the nature of the solution desired for the Mach waves associated with the layers of the shear zone. If $V_c < 0$, the Mach waves move to the left and when observed in the y-direction in $y > 1$, they appear to move outwards, so that we require $W > 0$. If $V_c > 0$ the waves are convected to the right and they appear to move inwards in $y > 1$, so that $W < 0$. Equation (A. 39) shows that these properties are associated with waves of the type (A. 37); the other possibility represents waves moving in the opposite directions (incident on the shear layer). Our solution must therefore be of the form (A. 37) in $y > 1$, so that from (A. 31)

$$\surd 2\,\alpha + (1+i)\,\beta = 0. \tag{A.40}$$

In the region $y < -1$, for waves of the type

$$\varpi(y) \propto \exp(i\lambda \int q\,dy), \tag{A.41}$$

equation (A. 38) is replaced by

$$W = \frac{nA(-\infty)}{\lambda(n-k_1)}, \tag{A. 42}$$

using (A. 13) for $y < -1$. Thus

$$W = \frac{-A(-\infty)}{(1+V_c^{-1})}, \tag{A. 43}$$

where $A(-\infty)$ is the dimensionless speed of sound far below the layer and is positive. When $V_c > 0$, reference to figure 5 shows that in $y < -1$ the Mach waves that we seek move outward so that $W < 0$, and when $V_c < 0$, $W > 0$; both statements being in accord with (A. 43). Thus, when $y < -1$, the waves generated by the shear zone are of the type (A. 41) and so from (A. 34),

$$\sqrt{2}\,\gamma + (1-i)\,\delta = 0. \tag{A. 44}$$

We now perform the matching of the solutions at $\xi = 0$. It will be recalled that $\eta(\xi)$ and $H(\xi)$ have the same value and the same derivative at $\xi = 0$, so that from (A. 8),

$$\varpi(y) = \frac{\{2\int q\,dy\}^{\frac{1}{4}}}{q^{\frac{1}{2}}}\,H(\xi) \tag{A. 45}$$

near $y = Y$. Thus

$$\left.\begin{aligned} \varpi(y) &\to \Omega^{-\frac{1}{4}}H(0) \quad \text{as} \quad y \to Y+0, \\ &\to -i\Omega^{-\frac{1}{4}}H(0) \quad \text{as} \quad y \to Y-0. \end{aligned}\right\} \tag{A. 46}$$

Now

$$H(\xi) = \frac{(-\frac{3}{4})!\,B}{2^{\frac{1}{4}}\pi} + \frac{\mu^{\frac{1}{4}}A}{2^{\frac{3}{4}}(\frac{1}{4})!}\,\xi + O(\tfrac{1}{2}\mu\xi^2)^{\frac{5}{4}} \tag{A. 47}$$

near $\xi = 0$ for branch I, and

$$H(\xi) = \frac{(-\frac{3}{4})!\,D}{2^{\frac{1}{4}}\pi} + \frac{\lambda^{\frac{1}{4}}C}{2^{\frac{3}{4}}(\frac{1}{4})!}\,\xi + O(\tfrac{1}{2}\lambda\xi^2)^{\frac{5}{4}}, \tag{A. 48}$$

for branch II. From (A. 46), then

$$B = -iD. \tag{A. 49}$$

Furthermore,

$$\frac{d\varpi}{dy} = H\frac{d}{dy}\{[2\int q\,dy]^{\frac{1}{4}}q^{-\frac{1}{2}}\} + [2\int q\,dy]^{-\frac{1}{4}}q^{\frac{1}{2}}\frac{dH}{d\xi}, \tag{A. 50}$$

from (A. 10), so that

$$\left.\begin{aligned} \frac{d\varpi}{dy} &\to \frac{\Omega^{\frac{1}{4}}\mu^{\frac{1}{4}}}{2^{\frac{3}{4}}(\frac{1}{4})!}A \quad \text{as} \quad y \to Y+0, \\ &\to \frac{i\Omega^{\frac{1}{4}}\lambda^{\frac{1}{4}}}{2^{\frac{3}{4}}(\frac{1}{4}\,!}C \quad \text{as} \quad y \to Y-0, \end{aligned}\right\} \tag{A. 51}$$

so that

$$\mu^{\frac{1}{4}}A = i\lambda^{\frac{1}{4}}C. \tag{A. 52}$$

Using (A. 27) and (A. 35) to replace $\alpha, \ldots, \delta$ in (A. 40) and (A. 44), the equations to be solved are

$$\left.\begin{aligned} \sqrt{2}\,A + (1+i)\,B &= \frac{\pi h(+0)}{2\sqrt{2}\mu}\{(1+i)\,\Phi_2(0) - \sqrt{2}\,\Phi_1(0)\}, \\ \sqrt{2}\,C + (1-i)\,D &= -\frac{i\pi h(+0)}{2\sqrt{2}\lambda}\{(1-i)\,\Phi_2(0) - \sqrt{2}\,\Phi_1(0)\}, \end{aligned}\right\} \tag{A. 53}$$

together with (A. 49) and (A. 52). But from (A. 22),

$$\Phi_1(0) = \int_0^\infty u^{-\frac{1}{4}} J_{-\frac{1}{4}}(u)\,du = \frac{(-\frac{3}{4})!}{2^{\frac{1}{4}}\pi^{\frac{1}{2}}},$$

$$\Phi_2(0) = \int_0^\infty u^{-\frac{1}{4}} J_{\frac{1}{4}}(u)\,du = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{4}}(-\frac{1}{4})!},$$

so that

$$\frac{\Phi_2}{\Phi_1} = \frac{\pi}{(-\frac{1}{4})!\,(-\frac{3}{4})!} = 2^{-\frac{1}{2}}.$$

If, therefore, for brevity, we write

$$\begin{aligned} L &= (\tfrac{1}{4}\pi)\,\Phi_1(0)\,h(+0) \\ &= 2^{-\frac{9}{4}}\pi^{\frac{1}{2}}(-\tfrac{3}{4})!\,h(+0), \end{aligned} \tag{A. 54}$$

the equations (A. 53) become

$$\left.\begin{aligned} \sqrt{2}\,A + (1+i)\,B &= -(1-i)\,\mu^{-1}L, \\ \sqrt{2}\,C + (1-i)\,D &= -(1-i)\,\lambda^{-1}L. \end{aligned}\right\} \tag{A. 55}$$

The solution to (A. 49), (A. 52) and (A. 55) is

$$\left.\begin{aligned} A &= -\frac{L}{\mu}\frac{1-i}{\sqrt{2}}\frac{\lambda-\mu}{\lambda+\mu}\left\{1-i\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}}\right\}, \\ B &= \frac{iL}{(\lambda\mu)^{\frac{1}{2}}}, \\ C &= \frac{L}{\lambda}\frac{1-i}{\sqrt{2}}\frac{\lambda-\mu}{\lambda+\mu}\left\{1-i\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}}\right\}, \\ D &= -\frac{L}{(\lambda\mu)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{A. 56}$$

and the combination of constants appearing in the asymptotic expression (A. 31) is

$$\begin{aligned} \sqrt{2}\,\alpha + (1-i)\,\beta &= -2i\beta \\ &= \frac{2L}{\mu}\left\{\left(\frac{\mu}{\lambda}\right)^2 + i\right\}. \end{aligned} \tag{A. 57}$$

If $M^2 \gg (k^2/Q_+^2)$, (k^2/Q_-^2) (for wave-numbers well inside the fan of figure 2), these expressions take the simpler forms

$$\left.\begin{aligned} A &= 0, \\ B &= \frac{iL}{M}, \\ C &= 0, \\ D &= -\frac{L}{M}, \\ \sqrt{2}\,\alpha + (1-i)\,\beta &= \frac{2L}{M}(1+i), \end{aligned}\right\} \tag{A. 58}$$

correct to $O(M^{-1})$.

REFERENCES

Chu, B. T. & Kovasznay, L. S. G. 1958 *J. Fluid Mech.* **3**, 494.

Erdélyi, A. 1956 *Asymptotic Expansions*. New York: Dover.

Howarth, L. 1953 Ed. *Modern Developments in Fluid Dynamics—High Speed Flow.* Oxford University Press.

Kramer, H. P. 1955 *J. Acoust. Soc. Amer.* **27**, 789.

Lassiter, L. W. & Heitkotter, R. H. 1954 *Nat. Adv. Comm. Aero.* (*Wash.*) *Tech. Note*, no. 3316.

Laufer, J. 1959 *Progr. Rep.* no. 20–378. J.P.L.: Calif. Inst. Tech.

Lighthill, M. J. 1952 *Proc. Roy. Soc.* A, **211**, 564.

Lighthill, M. J. 1954 *Proc. Roy. Soc.* A, **222**, 1.

Lighthill, M. J. 1956 Article in *Surveys in Mechanics*. Ed. G. K. Batchelor & R. M. Davies. Cambridge University Press.

Lighthill, M. J. 1958 *Fourier Analysis and Generalized Functions*. Cambridge University Press.

Lilley, G. M. 1958 *Aero. Res. Coun.* (*Lond.*), *Rep.* no. 20,376; F.M. 2724.

Phillips, O. M. 1957 *J. Fluid Mech.* **2**, 417.

Ribner, H. S. 1958 *Univ. Toronto Inst. Aerophys. Tech. Note*, no. 21.

Sanders, N. D. & Callaghan, E. E. 1956 *Proc. 2nd Int. Congr. Acoust.* Cambridge, Mass.

Townsend, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.